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STRESSES IN FIBERS SPANNING AN INFINITE MATRIX CRACK

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Abstract—The stresses in a unidirectional fiber composite subjected to tension parallel to the fibers, which has a single matrix crack bridged by fibers, are analyzed. Of interest is whether the enhanced load sustained by the fibers causes them to fail. In particular, we attempt to gain insight into the effect of the fiber–matrix interface on the stresses experienced by the fibers. As a preliminary approach to this issue, the analogous two-dimensional problem is studied. To simulate the influence of a real interface, we allow for interfacial slippage governed by a Coulomb friction law. Results for the dependence of the fiber stress on the fiber volume fraction, the interface parameters, and the load are presented.

INTRODUCTION

In ceramic–matrix composites, matrix cracking occurs to a greater or lesser extent. Under a tensile load applied parallel to the fibers, it is possible for a single crack in the matrix to traverse the entire specimen leaving the fibers intact. If the stresses sustained by these bridging fibers are not too high, then the applied load can be raised to the point that matrix cracks parallel to the first one can appear (Aveston *et al.*, 1971). On the other hand, if the stresses imposed on the fibers are too intense, then the matrix crack may penetrate the fibers causing composite failure. Hence, further matrix cracking and, consequently, the ultimate strength are dependent on the state of stress prevailing once a single bridged matrix crack has traversed the entire specimen. This suggests that a stress analysis of the configuration shown in Fig. 1 would be quite useful.

However, the problem that ought to be solved is even more complicated than an initial glance at Fig. 1 would suggest. Specifically, the character of the fiber–matrix interface plays an important role which must be incorporated. It is widely believed, at least among materials scientists dealing with ceramic–matrix composites, that matrix cracks are more likely to propagate into the fiber in systems that are well bonded. On the other hand, when the fiber–matrix interface is weak, matrix cracks tend to spare the fibers. A model which can quantify this effect of an imperfectly bonded interface has been put forward in a recent paper by

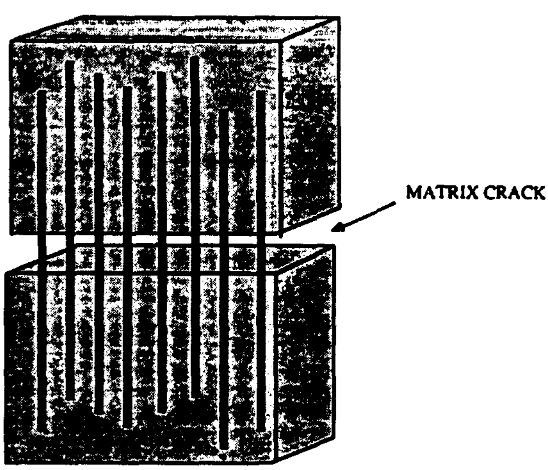

Fig. 1. Schematic of fibers spanning a matrix crack.

Dollar and Steif (1989), who focused on a single crack whose tips are impinging upon interfaces that are capable of frictional slip. They found that the stress at the crack tip was finite, and that the stress concentration was greater for interfaces that are "stronger", that is, more resistant to slip. While the Coulomb friction model used by Dollar and Steif (1989) is by no means the final word on descriptions of the interface, it is a tractable model that may give realistic trends for the dependence of the stress concentration on interface conditions.

In this paper we consider a problem which is the two-dimensional analog of the one depicted in Fig. 1. An infinite, two-dimensional solid, composed of alternating layers of fibers and matrix that are linked by Coulomb friction, is subjected to remote plane strain tension. Along the x-axis the matrix material is cracked. One expects that slippage occurs along the fiber–matrix interface in the vicinity of the matrix crack, and that remote from the crack plane the constituents stick to one another (there being no shear stress to drive slip). Below, we propose an approximate, yet accurate, means for solving this problem. This two-dimensional problem is considered because it can give at least qualitative insight into the potential effect of the interface on the fracture of the bridging fibers. Furthermore, the approximate method proposed here to solve this problem may ultimately be generalizable to the three-dimensional problem.

ANALYSIS

The two-dimensional problem considered here is shown schematically in Fig. 2. A remote stress σ_r is applied parallel to the fibers. For simplicity, the linear elastic fiber and matrix are taken to have the same isotropic moduli G and v. This seems justified for ceramic-matrix composites, where the fiber and matrix typically have similar moduli. To complete the description of this problem, we elaborate upon the interface law. Relative motion at the interface is modeled with Coulomb friction, an approach which has been used by the authors in two recent papers (Dollar and Steif, 1988, 1989). According to this interface law, each point along the interface is either sticking, slipping, or opening. For mathematical convenience, let the interface lie along the y-axis. Then, these three states are described as follows:

$$\text{stick condition} \qquad \sigma < 0, \quad |\tau| < \mu|\sigma|, \quad \frac{\mathrm{d}g}{\mathrm{d}t} = 0, \qquad h = \frac{\mathrm{d}h}{\mathrm{d}t} = 0 \tag{1a}$$

$$\text{slip condition} \quad \sigma < 0, \quad |\tau| = \mu|\sigma|, \quad \operatorname{sgn}\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right) = \operatorname{sgn}(\tau), \quad h = \frac{\mathrm{d}h}{\mathrm{d}t} = 0 \tag{1b}$$

$$\text{open condition} \qquad \sigma = \tau = 0, \qquad h > 0 \tag{1c}$$

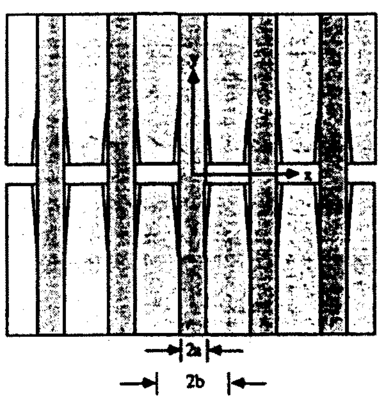

Fig. 2. Two-dimensional problem of frictionally constrained fibers spanning a matrix crack.

with

$$\sigma = \sigma_{xx} \quad \tau = \sigma_{xy}$$

$$g = \lim_{\varepsilon \to 0^+} [v(\varepsilon, y) - v(-\varepsilon, y)]$$

$$h = \lim_{\varepsilon \to 0^+} [u(\varepsilon, y) - u(-\varepsilon, y)].$$

In these equations σ_{xx} and σ_{xy} denote the usual Cartesian components of stress, u and v denote the x- and y-components of displacement, respectively, and μ is the friction coefficient. In applying eqns (1), one must be careful to use the total stresses, including any residual stresses. Thus, two parameters characterize each point on the interface: the residual stress (always normal if arising from, say, differences in thermal expansion coefficient) and the coefficient of friction. In the problem posed below, both quantities will be assumed to be constant along the interfaces. In particular, we simulate a residual stress at the interface by applying a uniform compressive stress σ_0. Note that this interface law assumes no bonding or adhesion at the interface.

Symmetry of the problem shown in Fig. 2 dictates that the extent of slip is the same for all fibers; however, it is dependent on the load in a manner which comes out of the analysis. It is also useful at the outset to bear in mind that all fibers carry the same average stress at the plane of the matrix crack, namely, σ_∞/V_f, where $V_f = a/b$ is the fiber volume fraction. While a finite element method could be used to solve this problem, as it takes advantage of the symmetry inherent in the infinite array of fibers, the resolution of stresses at the crack tip impinging upon the frictional interface may be unreliable. Instead, a method is offered here which is likely to be more accurate than would be a finite element method, but which is still relatively tractable.

To see the motivation for this alternative method, consider the consequences of using superposition and solving the equivalent problem of pressuring open the crack faces between the bridging fibers. Note further that this problem can be restated as one on a half-plane (see Fig. 3); the surface of the half-plane is free of shear stress and there are alternating regions of uniform pressure σ_∞ and zero normal displacement (where the fibers are). Note that the periodicity implies that the pressure along, say, $-b < x < -a$ is equilibrated by a tension along $-a < x < 0$.

Our method of solution is based on the following assumption: the stress field in the region where the matrix crack impinges upon the fiber that lies along $-a < x < a$ may be calculated—with satisfactory accuracy—by considering *only* the surface loading along $-b < x < b$. *This* loading of the half-plane, with the surface free of traction except along $-b < x < b$, is equivalent to an infinite medium with two semi-infinite cracks, one extending from $-\infty < x < -a$ and one extending from $a < x < \infty$, which are pressured open by σ_∞ along $-b < x < -a$ and $a < x < b$ (see Fig. 4). We base the above assumption upon the notion that the effects of any one of the self-equilibrating loadings of the boundary of

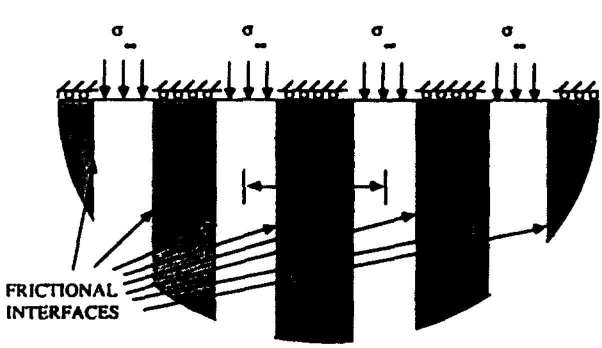

Fig. 3. Restatement of two-dimensional problem in terms of a half-plane.

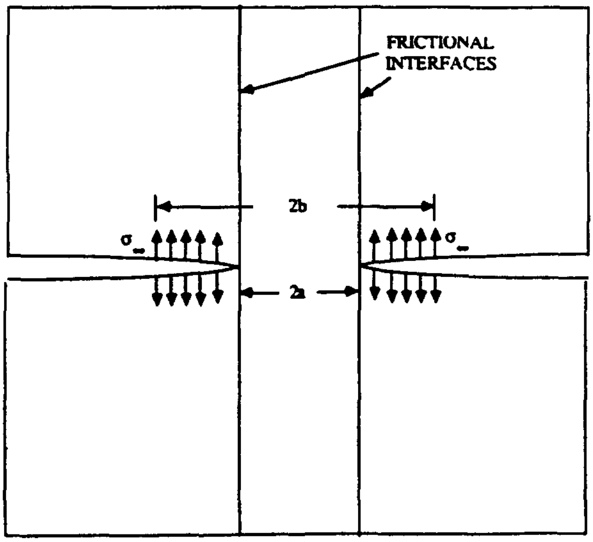

Fig. 4. Boundary value problem which mimics the infinite array of fibers spanning a matrix crack.

the half-plane (e.g., $nb-a < x < nb+a$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$) decay quickly with distance; in particular, the stress contribution associated with each self-equilibrating loading decays as $1/r^2$ for a perfectly bonded interface. In essence, this approximation neglects the effects of all but the loading closest to any particular fiber. In addition, the approximation neglects the influence of slippage along interfaces other than those bounding the fiber in question.

We offer further justification for this approach by applying it to the situation in which no slip is permitted at the fiber-matrix interface (perfect bonding); then, the problem we wish to solve is that of an infinite, periodic array of cracks, which has a closed form solution (Tada *et al.*, 1973). Our approximate method yields the problem shown in Fig. 4, but with no slippage at the interface. This also has a closed form solution which can be compared directly with the exact solution for the periodic array.

The solutions for these two problems are conveniently expressed in terms of the Muskhelishvili (1963) complex analytic functions ϕ and ψ, which are related to the stress and displacements according to

$$\sigma_{xx}+\sigma_{yy} = 2(\phi'+\overline{\phi'}) \tag{2a}$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy} = 2(\bar{z}\phi''+\psi') \tag{2b}$$

$$2G(u+iv) = \kappa\phi-z\overline{\phi'}-\bar{\psi} \tag{2c}$$

where $z = x+iy$, $(\ )'$ denotes complex differentiation with respect to z, an overbar denotes complex conjugation, and $\kappa = 3-4\nu$ in plane strain.

For the problem depicted in Fig. 4, one can obtain the solution ϕ—which we write as ϕ_0 for this case of no interfacial slippage—by a standard method (conversion to a Hilbert problem and integration of a Cauchy singular integral). ϕ_0 is found to be

$$\phi_0' = \frac{-\sigma_\infty}{\pi i}\left\{\frac{1}{2}\log\left[\frac{\sqrt{z^2-a^2}-\sqrt{b^2-a^2}}{\sqrt{z^2-a^2}+\sqrt{b^2-a^2}}\right]+\frac{\sqrt{b^2-a^2}}{\sqrt{z^2-a^2}}\right\}. \tag{3}$$

For this class of problems, the other potential, ψ_0, is related to ϕ_0 by

$$\psi_0' = -z\phi_0''. \tag{4}$$

The exact solution for the periodic array of cracks (each of half-length $c \equiv b-a$) is given by

$$\phi' = \tfrac{1}{2}\sigma_x \left\{ \frac{\sin \dfrac{\pi z}{2b}}{\sqrt{\sin^2 \dfrac{\pi z}{2b} - \sin^2 \dfrac{\pi c}{2b}}} - 1 \right\}. \tag{5}$$

Various aspects of the two solutions just given may be compared. For example, the respective stress intensity factors are

$$K_{ex} = \sigma_x \sqrt{2b \tan \frac{\pi c}{2b}} \quad K_{app} = 2\sigma_x \sqrt{\frac{b^2 - a^2}{\pi a}} \tag{6a,b}$$

where K_{ex} and K_{app} denote the exact (periodic cracks) and approximate (Fig. 4) stress intensity factors, respectively. At worst, when $b-a \ll a$ (isolated cracks or closely spaced fibers), these solutions differ by 11%. When $b \gg a$ (nearly impinging cracks or widely spaced fibers), the stress intensity factors approach the same value

$$K = \frac{2\sigma_x b}{\sqrt{\pi a}}. \tag{7}$$

It is not surprising that our approximate method yields accurate results for widely spaced fibers; in this limit the fibers that transmit the applied stress across the matrix crack plane do not see one another. However, it is remarkable how close the stress intensity factors are for typically spaced fibers. For example, with a fiber volume fraction of 50% ($a/b = 0.5$), the stress intensity factors differ by 2.3%. The reasonable accuracy obtained in applying this approximate method to the case of perfect bonding is taken as sufficient justification for using this method when slippage occurs at the fiber–matrix interface. It ought to be mentioned, however, that one should expect our method to be somewhat less accurate for the case of slipping fibers, in which the rates of decay will be slower (see Dollar and Steif, 1988).

Before we outline the solution method for the case of slippage at the interface, we consider briefly the nature of the fields in the vicinity of the crack tips. The near-tip behavior when a crack impinges upon a slipping interface was discussed in some detail in a previous paper by the authors (Dollar and Steif, 1989); we found that the stress at the crack tip was finite. That is, there are no admissible crack-tip eigenfunctions, satisfying the conditions of traction-free crack faces and frictional slippage at the interface, which exhibit singular stresses. The dominant eigenfunction is one involving piece-wise constant stresses, the only non-zero stress component being the tensile stress σ_{yy} ahead of the crack tip. Our solution method is such that this near-tip behavior can be simulated. [Actually, the additional condition imposed in Dollar and Steif (1989) was that slippage was to occur such that the crack opened; for a closing crack, a singular compressive stress ahead of the crack tip is possible.]

The solution method follows closely the method used in other papers focusing on the effects of frictional slippage (Dollar and Steif, 1988, 1989). Slippage at the interface is represented by a continuous distribution of dislocations. The total stresses are the sums of the stresses associated with the perfectly bonded solution (ϕ_0, ψ_0) and the stresses associated with the distributed dislocations. One obtains a singular integral equation for the dislocation density by enforcing the friction condition (1b) along the slip zone. The length of the slip zone is unknown and is found as part of the solution.

To use superposition as indicated above, one must have, as the kernel solution, the solution to the problem of a dislocation in an infinite medium which has two semi-infinite, traction-free cracks. The technique to find such a solution is given by Lo (1978) (among others), who used distributed dislocations to represent the kinked part of a kinked crack. For a dislocation with Burger's vector $\mathbf{b} = b_y \mathbf{j}$ one finds this kernel solution to be given by

$$\phi' = \phi'_\infty + \phi'_R \tag{8a}$$

$$\psi' = \psi'_\infty + \psi'_R \tag{8b}$$

where

$$\phi'_\infty(z, z_0) = \frac{\alpha}{z - z_0} \quad \psi'_\infty = \frac{\alpha}{z - z_0} - \frac{\alpha \bar{z}_0}{(z - z_0)^2} \tag{9a,b}$$

$$\phi'_R(z, z_0) = -\alpha[F(z, z_0) + F(z, \bar{z}_0) + (z_0 - \bar{z}_0)H(z, \bar{z}_0)] + \frac{c}{X(z)} \tag{10}$$

$$\psi'_R(z, z_0) = \bar{\phi}'_R(z, z_0) - \phi'_R(z, z_0) - z\phi''_R(z, z_0) \tag{11}$$

$$F(z, z_0) = \frac{1 - \dfrac{X(z_0)}{X(z)}}{2(z - z_0)} \tag{12}$$

$$H(z, z_0) = \frac{\partial F(z, z_0)}{\partial z_0} \tag{13}$$

$$X(z) = \sqrt{z^2 - a^2} \tag{14}$$

$$\alpha = \frac{G b_y}{\pi(\kappa + 1)}. \tag{15}$$

This solutions appears to be quite similar to the kernel solution for a finite crack (see Lo, 1978). One important difference here is that the branch of the square root $\sqrt{z^2 - a^2}$ is the one which has discontinuities along the branch cuts $-\infty < x < -a$ and $a < x < \infty$. Here, the constant c can be evaluated by noting that the function $c/\sqrt{z^2 - a^2}$ is the solution to the problem of a flat, rigid, frictionless punch applied to the lower (or upper) half-plane. It is obvious that a solution to the punch problem (with an arbitrary load) may be superposed on the dislocation solution. Clearly, it is desired to have that dislocation solution which involves zero net force transmitted across $-a < x < a$. Hence, c must be zero.

To formulate an integral equation, we now assume that the interface remains closed everywhere and that slip occurs along a single portion of each interface ($x = \pm a$, $-L < y < L$), with the remainder of the interface being in a stick condition. [One must determine *a posteriori* if, in fact, the interface remains closed; this is done by checking the results to see whether $\sigma_{xx} < 0$. For the results presented, including friction coefficients up to $\mu = 0.3$, the interface was found to remain closed. Furthermore, since the interfacial compression increases with the applied load, it is suspected that the interface remains closed for all levels of load. On the other hand, as found in Dollar and Steif (1989), there is some opening at the interface for higher friction coefficients, provided the applied load is sufficiently small.] Then, it is necessary to distribute dislocations (with Burger's vector in the y-direction) only along the slipped portions of the interface. The distribution of dislocations is chosen to satisfy the following integral equation which enforces the friction condition $\sigma_{xy} = \pm\mu|\sigma_{xx}|$ along the slip zones:

$$\int_0^L b(y_0)[R_0(y,y_0)+R_1(y,y_0)+\mu R_2(y,y_0)]\,\mathrm{d}y_0+f(y)=0 \tag{16}$$

where $b(y)$ is the dislocation density, and the functions R_0 (the singular part), R_1, R_2 and f are given in the Appendix.

Of the various quantities which may be computed from the solution, the tensile stress immediately ahead of the crack tips is the most important one considered here. This tensile stress, $(\sigma_{yy})_{\text{tip}}$, can be obtained in two ways. First, it is readily shown that this stress is related to the dislocation density as one approaches the crack tip from the slip zone according to

$$(\sigma_{yy})_{\text{tip}}=\frac{E}{1-\nu^2}b(0). \tag{17}$$

Alternatively, the stress at various points ahead of the crack tip can be computed from the entire distribution of dislocations, followed by an extrapolation to the crack tip. The degree to which these two methods yield the same number is a measure of the accuracy of the numerical solution to (16). Generally, agreement to within a few percent was found.

The results to be presented will indicate how the tensile stress at the crack tips, and, hence, the likelihood of fiber failure before multiple cracking can set in, depends on conditions at the interface. In addition, we will present results for the extent of slip, the crack-tip opening (maximum slippage at the interface), and the rate of load transfer from the fiber to the matrix.

RESULTS

In this section we present results which indicate the dependence of quantities of interest on the material parameters. It is simplest to consider first the limiting situation of a very low concentration of fibers. In such circumstances, one can model the problem as a single fiber being pulled out of a half-plane. This problem, shown schematically in Fig. 5, is similar to an earlier problem considered by the authors (Dollar and Steif, 1988) in which a fiber is pulled out from (or pushed into) a half-plane by a uniform normal stress. (A by-product of considering this limiting problem is the opportunity to contrast the two boundary conditions of a uniform displacement and a uniform traction). With this limiting case, one can see the essential effect of the load and the friction coefficient. Later, when the full problem is considered, the effect of nearby fibers will be evident.

When the interface is perfectly bonded, the solution to the fiber pull-out problem is the negative of the solution for a rigid, flat, frictionless punch pressed into a half-plane. Therefore, the solution, ϕ_p', is given by (Muskhelishvili, 1963)

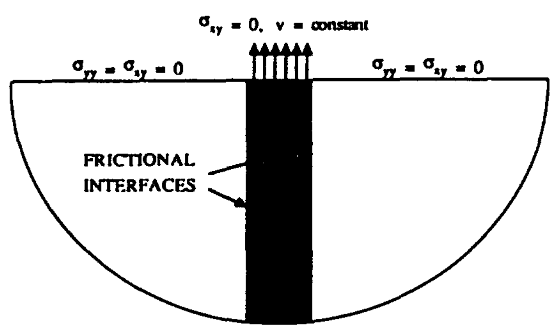

Fig. 5. Pull-out problem which is equivalent to limiting case of $b/a \to \infty$.

$$\phi_p' = \frac{iP}{2\pi X(z)} \tag{18}$$

where P is the load transmitted through the fiber and $X(z)$ is defined by (14). As expected, the stresses are square-root singular as the crack tip is approached. Once we allow frictional slip to occur, however, the interface serves to "blunt" the impinging matrix crack. To see this effect, consider Fig. 6 in which the stress at the tip, normalized by the average fiber load is plotted as a function of the fiber load. The dependence on the fiber load, insofar as it is normalized by $\mu\sigma_0$, is typical of problems involving frictional interfaces. For loads that are small compared with the nominal friction stress $\mu\sigma_0$, one, in some sense, recovers the perfectly bonded case. As the applied stress increases relative to $\mu\sigma_0$, the blunting effect of the interface increases.

[Actually, the perfectly bonded case is not precisely recovered for small loads. In that limit, our problem becomes a small scale slipping problem (studied by Dollar and Steif, 1989), in which the crack tip in effect senses a remotely applied elastic singular field. As shown by Dollar and Steif (1989), the tensile stress at the crack tip in the small scale slipping limit is actually proportional to the residual stress σ_0. Since this is the case for arbitrarily small applied loads, the corresponding stress *concentration* factor properly becomes infinite in the limit of a vanishing small load.]

The dependence on μ is not solely through the combination $\mu\sigma_0$, however. This dependence was an important aspect of a recent theoretical study of the pull-out test by Dollar and Steif (1988). They compared results based on the Coulomb friction interface model with results based on a well-established approximate method of analysis. This method, which can be used to compute the extent of slip, the relative displacement at the interface and the load transfer, rests on the assumption of a constant shear stress prevailing at the interface. (For convenience, we refer to this method as the CSSA —the constant shear stress approximation.) Comparisons were made to see whether it is sufficient to employ a CSSA (in which $\mu\sigma_0$ is taken to be the constant shear stress), even through the friction stress, strictly speaking, varies point-wise according to the Coulomb friction law. To the extent that the Coulomb friction-based results are dependent only on the product $\mu\sigma_0$, and not on μ and σ_0 individually, the CSSA is adequate. In fact, the CSSA was found in some instances to be adequate, particularly when the friction coefficient is small. Unfortunately, there appears not be a CSSA to the stress concentration at the crack tip; thus, no such comparison is possible. In any event, a CSSA could not possibly predict the stress concentration accurately since it clearly depends on μ and σ_0 individually, and not just upon $\mu\sigma_0$.

Turning to Fig. 7, one can see the slip length (still for the case of widely spaced fibers). Results obtained here are plotted as the solid lines; two other results are shown for

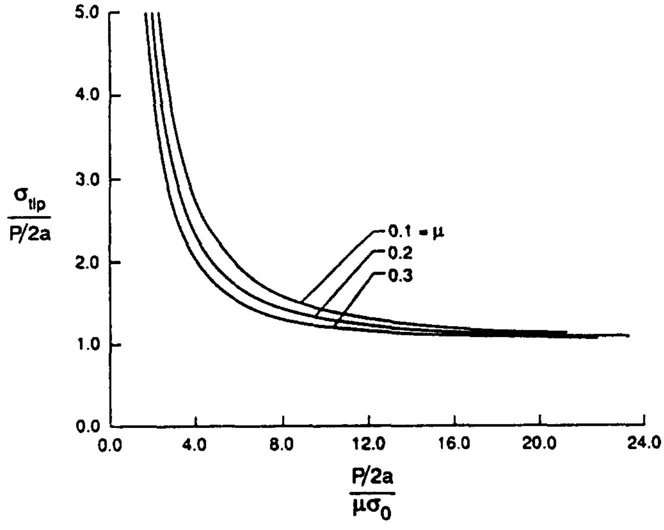

Fig. 6. Tensile stress at crack tip as a function of pull-out load.

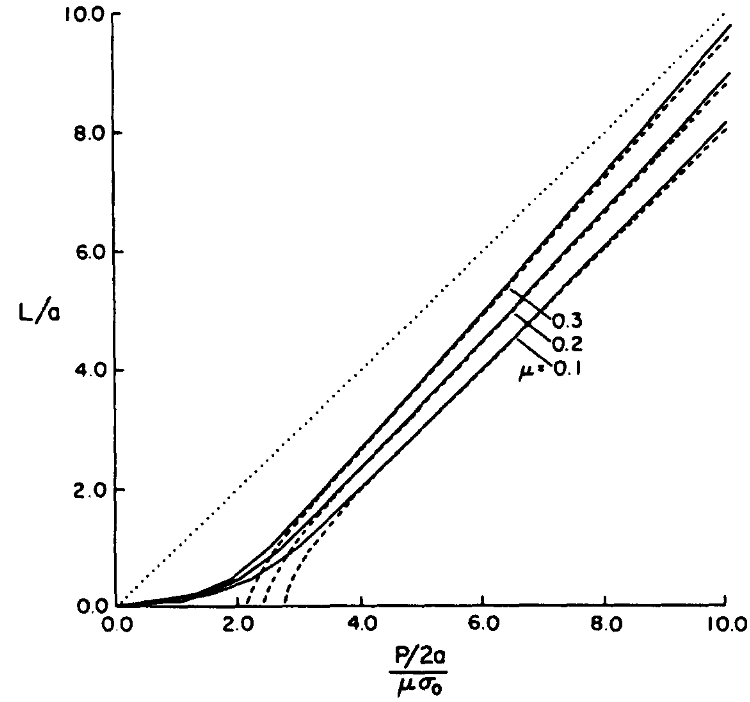

Fig. 7. Slip length as a function of pull-out load. (——) Present problem of uniform applied displacement; (- - -) uniform applied traction; (· · ·) constant shear stress approximation.

comparison. The dotted line is the prediction based on the CSSA, which is described in more detail below for the particular problem considered here. Clearly, there is some discrepancy, particularly for low loads. With increasing loads, however, the CSSA may be viewed as adequate. Also shown in Fig. 7 are the results of a previous analysis of the pull-out problem (Dollar and Steif, 1988), in which a uniform normal tension was applied to the fiber (see dashed curves). Again, there are discrepancies at low loads. Under a uniform tension, slip does not initiate immediately because the shear stress is less than μ times the normal stress. Under a uniform displacement (the crack problem here), slippage occurs immediately upon application of the load. However, the curves rapidly approach one another as the load is increased. In fact, in the limit of a very large load, the two problems become identical; the tractions across the fiber (to which a uniform displacement is applied) become uniform, as suggested by the stress concentration shown in Fig. 6. (The slight discrepancies between the solid and dashed curves in Fig. 7 for large loads are associated with slightly different numerical formulations.)

Figure 8 indicates the crack tip opening displacement. This opening corresponds to the maximum amount of slip at the interface (as $y \to 0$). As in Fig. 7, the solid lines are the results of the calculations carried out here; the dotted line is the CSSA and the dashed lines are the results of the pull-out test with a uniform tension. The CSSA agrees reasonably well with the present results, though it, of course, does not predict any dependence on μ alone. As mentioned above, interfacial slippage does not occur in the uniform tension pull-out test until a finite load is applied. For higher loads there appears to be a constant discrepancy between the present results and the uniform tension pull-out test.

We now allow the matrix crack to be spanned by a *non-dilute* concentration of fibers (finite values of b/a). First, the dependence of the stress concentration on the load is shown in Fig. 9 for various b/a ($\mu = 0.3$). As before, the stress concentration diminishes with the dimensionless applied stress. To allow a proper comparison for a range of b/a, we have defined the dimensionless stress to be the average stress transmitted by the fiber when the remote load is applied, normalized by the nominal friction stress $\mu\sigma_0$. One can see that the stress concentration increases with the fiber volume fraction, assuming a fixed amount of load transmitted by the fiber.

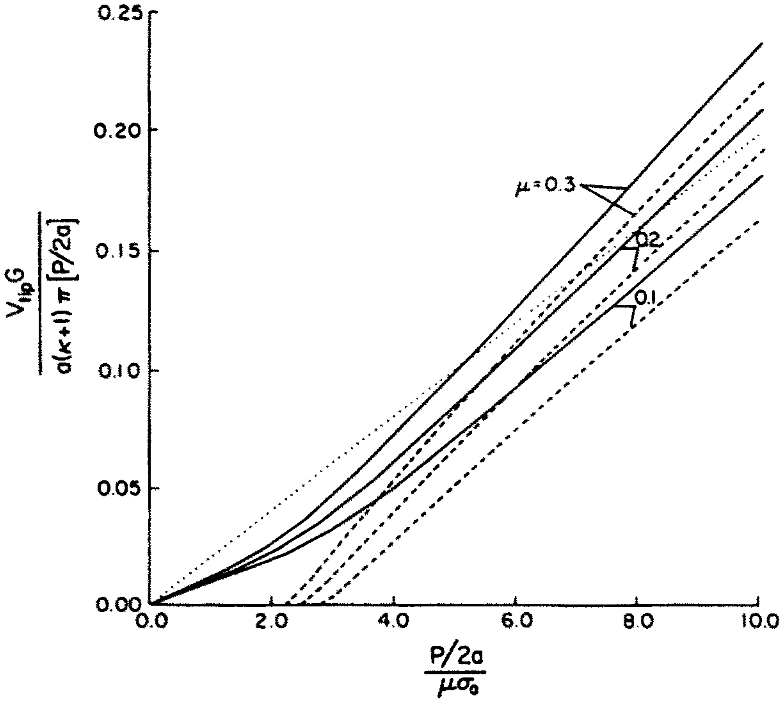

Fig. 8. Crack tip opening as a function of pullout load. (——) Present problem of uniform applied displacement; (- - -) previous problem of uniform applied traction; (· · ·) constant shear stress approximation.

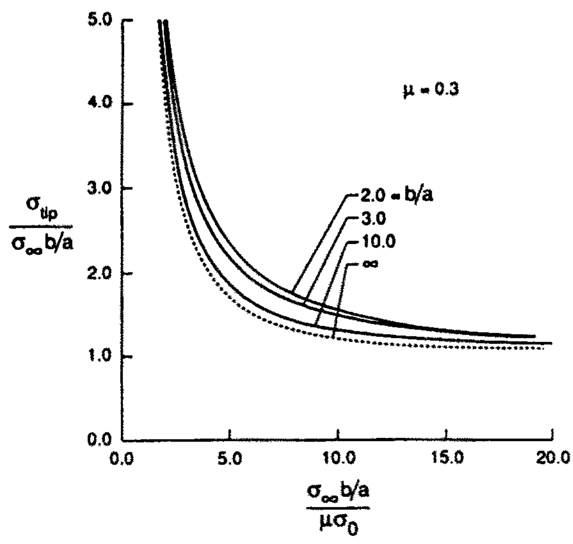

Fig. 9. Tensile stress at crack tip as a function of opening pressure.

The extent of slip is plotted in Fig. 10 as a function of the dimensionless load. As indicated above, the slip length is one of several quantities that may be computed on the basis of a CSSA. Let $\mu\sigma_0$ be the constant shear stress acting across the interface. The slip length is taken to be the distance over which all of the load in the fiber is transferred to the matrix. This approximate slip length, L_{CSSA}, which is readily determined to be

$$\frac{L_{CSSA}}{a} = \frac{\sigma_\infty}{\mu\sigma_0}\frac{b}{a}\left[1-\frac{a}{b}\right] \tag{19}$$

is plotted in Fig. 10 as the dotted line. As can be seen, the approximation may be satisfactory

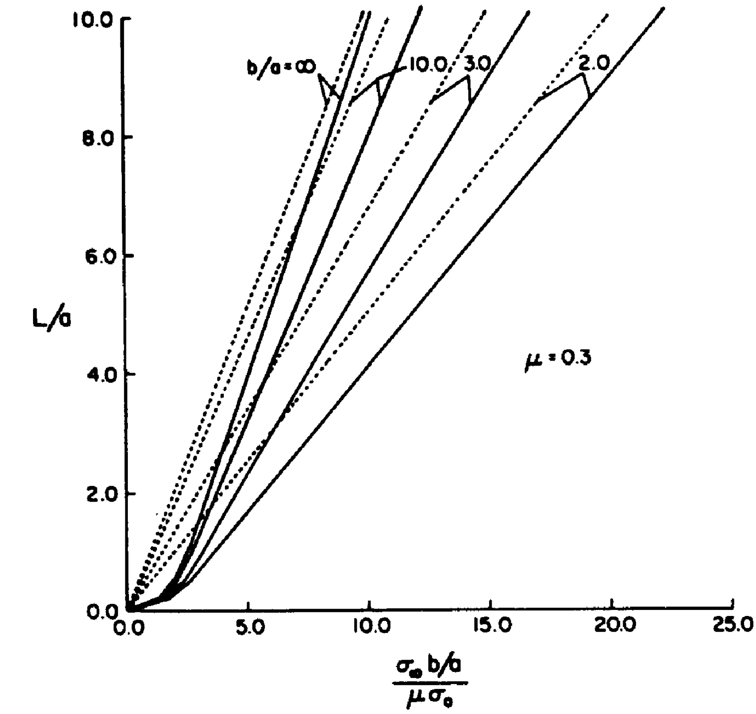

Fig. 10. Slip length as a function of opening pressure. (——) Numerical solution to integral equation; (– – –) constant shear stress approximation.

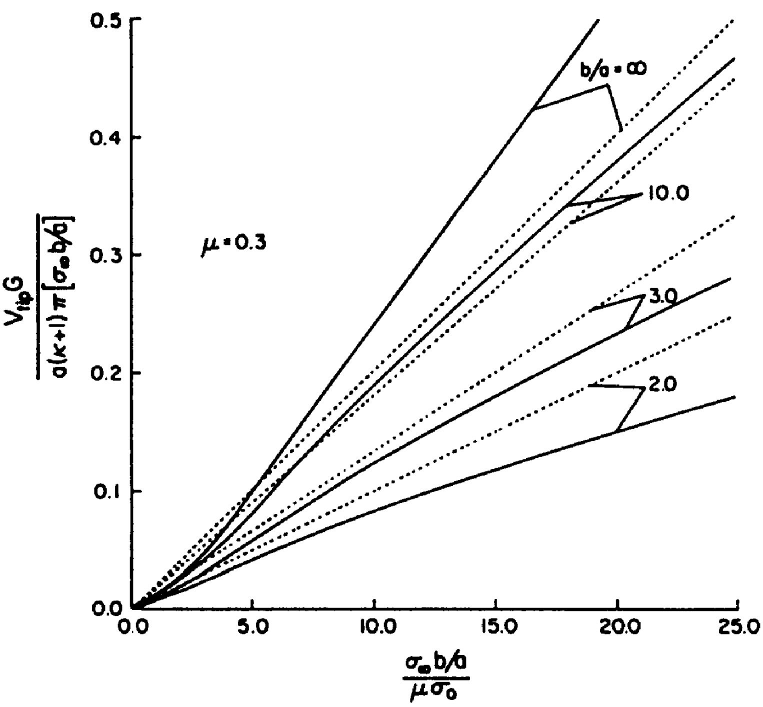

Fig. 11. Crack tip opening as a function of opening pressure. (——) Numerical solution to integral equation; (– – –) constant shear stress approximation.

depending on the desired accuracy. Note that the limit of a dilute concentration of fibers is obtained by fixing $\sigma_\infty b/a$ and letting $a/b \to 0$.

In Fig. 11 the crack tip opening is plotted as a function of the dimensionless load. For comparison, the CSSA to the slippage at the interface is shown as the dotted lines. The

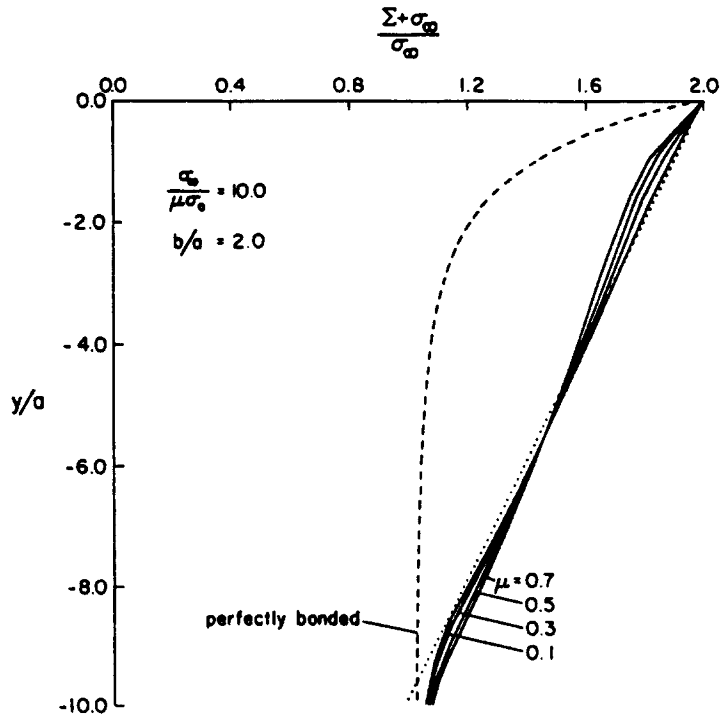

Fig. 12. Average fiber stress as a function of distance away from matrix crack. (——) Numerical solution to integral equation; (---) perfectly bonded interface; (···) constant shear stress approximation.

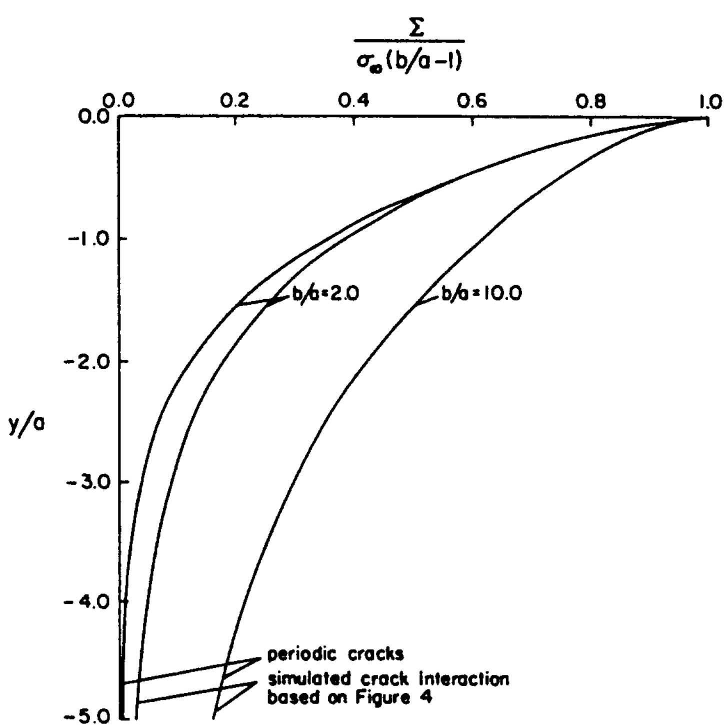

Fig. 13. Comparison of load transfer results for actual infinite array of fibers and for simulated fiber interaction.

approximate slippage is readily derived by integrating the difference (across the interface) in the longitudinal strain, the integration being from the matrix crack to the end of the slip zone. The result is

$$\frac{Gv_{\text{CSSA}}}{a(\kappa+1)\pi\sigma_x b/a} = \frac{1}{16\pi}\frac{\sigma_x}{\mu\sigma_0}\frac{b}{a}\left(1-\frac{a}{b}\right). \tag{20}$$

In particular, for a fiber volume fraction of 0.5 ($b/a = 2$), the CSSA given by (20) differs from the numerical solution of the integral equation by up to 40%, at least for a friction coefficient of 0.3. Note also that earlier work on the pull-out test would suggest that the CSSA and numerical results based on a Coulomb friction interface deviate with increasing friction coefficient. Although it must be borne in mind that the present analysis is approximate, particularly insofar as the configuration of Fig. 4 is analyzed, the results still suggest that the CSSA be applied cautiously to computing the opening at the crack tip.

We now turn to the transfer of load from the fibers to the matrix, a crucial element in many theories of composite strength. Consider Fig. 12, in which the average stress in the fiber, normalized by the average stress at the matrix–crack plane, is plotted as a function of distance from the matrix–crack plane. Different solid curves correspond to different coefficients of friction (b/a is fixed at 2.0). The CSSA, which predicts linear load transfer, is plotted as the dotted line in Fig. 12. For comparison, the load transfer rate assuming perfect bonding is also plotted (dashed line). As a previous study indicated (Dollar and Steif, 1988), load transfer with Coulomb friction is generally substantially slower than under conditions of perfect bonding. [When a fiber is being pushed in Dollar and Steif (1988), the predictions of perfect bonding and a Coulomb friction model approach one another as the friction coefficient becomes very large.] Here, the CSSA is in fair agreement with the Coulomb friction results.

Finally, we provide an additional means of gauging the accuracy of the approach that we have taken to model the infinite array of fibers. By consideration of the stress intensity factors for the perfectly bonded problems, it was shown above that our method yields results that are a few percent different from the exact result for an infinite line of cracks, at least for typical fiber volume fractions. This gives confidence to our predictions for the stress concentration at the crack tip when slippage occurs. In Fig. 13, we compare the load transfer rates for these two perfectly bonded problems. The agreement between them is quite good when the fibers are relatively far apart, consistent with the original justification offered above for our approach. Even with typical fiber volume fractions ($b/a = 2.0$), the solutions are similar near the matrix crack; substantial deviations are evident further from the matrix crack. It is more difficult to say anything certain about the accuracy of the solution once slip occurs. However, the agreement between the CSSA and our results leads one to have some faith in the load transfer results, at least near the matrix crack.

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APPENDIX

In this Appendix we give expressions for the terms in the dimensionless version of eqn (16). Spatial variables have been non-dimensionalized by a, stresses by σ_∞ or $P/2a$ as appropriate, and the dislocation intensity $b(y_0)$ by $P\pi(\kappa+1)/(2Ga)$ or $\sigma_\infty\pi(\kappa+1)/G$ as appropriate.

$$R_0(y,y_0) = \frac{-2}{y-y_0}$$

$$R_1(y,y_0) = 2\,\mathrm{Im}\,[h_2] + \mathrm{Im}\left[\frac{\bar h_3^2+h_3^2}{\bar h_3} + \frac{\bar h_4^2+h_4^2}{\bar h_4}\right] + 2y\,\mathrm{Re}\,[H'(z,z_0)]$$

$$R_2(y,y_0) = -\mathrm{Re}\left[\frac{\bar h_3^2+h_3^2}{\bar h_3} + \frac{\bar h_4^2+h_4^2}{\bar h_4}\right] + 2y\,\mathrm{Im}\,[H'(z,z_0)] - 2\,\mathrm{Re}\,[H(z,z_0)]$$

where

$$h_2 = \frac{1}{z-\bar z_0},\quad h_3 = \frac{1}{z+z_0},\quad h_4 = \frac{1}{z+\bar z_0},$$

$$H(z,z_0) = \frac{2}{[X(z)+X(z_0)]^2}\left[1+\frac{z_0^3-1}{X(z)X(z_0)}\right] + \frac{2}{[X(z)-\overline{X(z_0)}]^2}\left[1-\frac{\bar z_0^3-1}{X(z)\overline{X(z_0)}}\right].$$

The function $f(y)$ is written as

$$f(y) = f_1(y) + \mu f_2(y)$$

where

$$f_1(y) = \frac{-2}{\pi}\,y\,\mathrm{Im}\left[\frac{z}{[X(z)]^3}\right]$$

$$f_2(y) = \frac{2}{\pi}\left\{-\mathrm{Im}\left[\frac{1}{X(z)}\right] + y\,\mathrm{Re}\left[\frac{z}{[X(z)]^3}\right]\right\} - \frac{\sigma_0}{P/2a};$$

for the pull-out problem ($b \to \infty$) and

$$f_1(y) = -2y\,\mathrm{Re}\,[\phi_0''(z)]$$

$$f_2(y) = 2\{\mathrm{Re}\,[\phi_0'(z)] - y\,\mathrm{Im}\,[\phi_0''(z)]\} - \frac{\sigma_0}{\sigma_\infty}$$

for the case of finite b. The function ϕ_0' is given by eqn (3).